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On LS-Category of a Family of Rational Elliptic Spaces II

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Abstract: Let X be a finite type simply connected rationally elliptic CW-complex with Sullivan minimal model $(\Lambda V, d)$ and let $k \geq 2$ the biggest integer such that $d = \sum_{i \geq k} d_i$ with $d_i(V) \subseteq \Lambda^i V$. If $(\Lambda V, d_k)$ is moreover elliptic then $\mathrm{cat}(\Lambda V, d) = \mathrm{cat}(\Lambda V, d_k) = \dim(V^{\mathrm{even}})(k-2) + \dim(V^{\mathrm{odd}})$. Our work aims to give an almost explicit formula of LS-category of such spaces in the case when $k \geq 3$ and when $(\Lambda V, d_k)$ is not necessarily elliptic.

Key words: Elliptic spaces, Lusternik-Schnirelman category, Toomer invariant.

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1. Introduction

The *Lusternik-Schirelmann category* (c.f. [7]), $\mathrm{cat}(X)$, of a topological space X is the least integer n such that X can be covered by $n+1$ open subsets of X, each contractible in X (or infinity if no such n exists). It is an homotopy invariant (c.f. [3]). For X a simply connected CW complex, the *rational L-S category*, $\mathrm{cat}_0(X)$, introduced by Félix and Halperin in [2] is given by $\mathrm{cat}_0(X) = \mathrm{cat}(X_{\mathbb{Q}}) \leq \mathrm{cat}(X)$.

In this paper, we assume that X is a simply connected topological space whose rational homology is finite dimensional in each degree. Such space has a Sullivan minimal model $(\Lambda V, d)$, i.e. a commutative differential graded algebra coding both its rational homology and homotopy (cf. §2).

By [1, Definition 5.22] the rational *Toomer invariant* of X, or equivalently of its Sullivan minimal model, denoted by $e_0(\Lambda V, d)$, is the largest integer s for which there is a non trivial cohomology class in $H^*(\Lambda V, d)$ represented by a cocycle in $\Lambda^{\geq s} V$, this coincides in fact with the Toomer invariant of the fundamental class of $(\Lambda V, d)$. As usual, $\Lambda^s V$ denotes the elements in ΛV of "*wordlength*" s. For more details [1], [3] and [14] are standard references.

In [4] Y. Felix, S. Halperin and J. M. Lemaire showed that for Poincaré duality spaces, the rational L-S category coincides with the rational Toomer invariant $e_0(X)$, and in [9] A. Murillo gave an expression of the fundamental class of $(\Lambda V, d)$ in the case where $(\Lambda V, d)$ is a pure model (cf. §2).

Let then $(\Lambda V, d)$ be a Sullivan minimal model. The differential d is decomposable, that is, $d = \sum_{i \geq k} d_i$, with $d_i(V) \subseteq \Lambda^i V$ and $k \geq 2$.

Recall first that in [8] the authors gave the explicit formula $\operatorname{cat}(\Lambda V, d) = \dim V^{\text{odd}} + (k-2)\dim V^{\text{even}}$ in the case when $(\Lambda V, d_k)$ is also elliptic.

The aim of this paper is to consider another class of elliptic spaces whose Sullivan minimal model $(\Lambda V, d)$ is such that $(\Lambda V, d_k)$ is not necessarily elliptic. To do this we filter this model by

$$F^p = \Lambda^{\geq (k-1)p} V = \bigoplus_{i=(k-1)p}^{\infty} \Lambda^i V. \tag{1}$$

This gives us the main tool in this work, that is the following convergent spectral sequence (cf. §3):

$$H^{p,q}(\Lambda V, \delta) \ \Rightarrow \ H^{p+q}(\Lambda V, d). \tag{2}$$

Notice first that, if $\dim(V) < \infty$ and $(\Lambda V, \delta)$ has finite dimensional cohomology, then $(\Lambda V, d)$ is elliptic. This gives a new family of rationally elliptic spaces.

In the first step, we shall treat the case under the hypothesis assuming that $H^N(\Lambda V, \delta)$ is one dimensional, being N the formal dimension of $(\Lambda V, d)$ (cf. [5]). For this, we will combine the method used in [8] and a spectral sequence argument using (2). We then focus on the case where $\dim H^N(\Lambda V, \delta) \geq 2$. Our first result reads:

Theorem 1. *If $(\Lambda V, d)$ is elliptic, $(\Lambda V, d_k)$ is not elliptic and $H^N(\Lambda V, \delta) = \mathbb{Q}.\alpha$ is one dimensional, then*

$$\operatorname{cat}_0(X) = \operatorname{cat}(\Lambda V, d) = \sup\{s \geq 0,\ \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^{\geq s} V\}.$$

Let us explain in what follow, the algorithm that gives the first inequality,

$$\operatorname{cat}(\Lambda V, d) \geq \sup\{s \geq 0,\ \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^{\geq s} V\} := r.$$

i) Initially we fix a representative $\omega_0 \in \Lambda^{\geq r} V$ of the fundamental class α with r being the largest s such that $\omega_0 \in \Lambda^{\geq s} V$.

ii) A straightforward calculation gives successively:

$$\omega_0 = \omega_0^0 + \omega_0^1 + \cdots + \omega_0^l$$

with

$$\omega_0^i = (\omega_0^{i,0}, \omega_0^{i,1}, \ldots, \omega_0^{i,k-2}) \in \Lambda^{(k-1)(p+i)}V \oplus \Lambda^{(k-1)(p+i)+1}V \oplus \cdots \oplus \Lambda^{(k-1)(p+i)+k-2}V.$$

Using $\delta(\omega_0) = 0$ we obtain $d\omega_0 = a_2^0 + a_3^0 + \cdots + a_{t+l}^0$ with

$$a_i^0 = (a_i^{0,0}, a_i^{0,1}, \ldots, a_i^{0,k-2}) \in \Lambda^{(k-1)(p+i)}V \oplus \Lambda^{(k-1)(p+i)+1}V \oplus \cdots \oplus \Lambda^{(k-1)(p+i)+k-2}V.$$

iii) We take t the largest integer satisfying the inequality:

$$t \leq \frac{1}{2(k-1)}\big(N - 2(k-1)(p+l) - 2k + 5\big).$$

Since $d^2 = 0$, it follows that $a_2^0 = \delta(b_2)$ for some

$$b_2 \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)-(k-1)+j}V.$$

iv) We continue with $\omega_1 = \omega_0 - b_2$.

v) By the imposition iii), the algorithm leads to a representative $\omega_{t+l-1} \in \Lambda^{\geq r}V$ of the fundamental class of $(\Lambda V, d)$ and then $e_0(\Lambda V, d) \geq r$.

Now, $\dim(V) < \infty$ imply $\dim H^N(\Lambda V, \delta) < \infty$. Notice also that the filtration (1) induces on cohomology a graduation such that $H^N(\Lambda V, \delta) = \oplus_{p+q=N} H^{p,q}(\Lambda V, \delta)$. There is then a basis $\{\alpha_1, ..., \alpha_m\}$ of $H^N(\Lambda V, \delta)$ with $\alpha_j \in H^{p_j,q_j}(\Lambda V, \delta)$, $(1 \leq j \leq m)$. Denote by $\omega_{0j} \in \Lambda^{\geq r_j}V$ a representative of the generating class α_j with r_j being the largest s_j such that $\omega_{0j} \in \Lambda^{\geq s_j}V$. Here p_j and q_j are filtration degrees and $r_j \in \{p_j(k-1), \ldots, p_j(k-1)+(k-2)\}$.

The second step in our program is given as follow:

Theorem 2. *If $(\Lambda V, d)$ is elliptic and $\dim H^N(\Lambda V, \delta) = m$ with basis $\{\alpha_1, \ldots, \alpha_m\}$, then, there exists a unique p_j, such that*

$$\mathrm{cat}_0(X) = \sup\{s \geq 0,\ \alpha_j = [\omega_{0j}] \text{ with } \omega_{0j} \in \Lambda^{\geq s}V\} := r_j.$$

Remark 1. The previous theorem gives us also an algorithm to determine LS-category of any elliptic Sullivan minimal model $(\Lambda V, d)$. Knowing the largest integer $k \geq 2$ such that $d = \sum_{i \geq k} d_i$ with $d_i(V) \subseteq \Lambda^i V$ and the formal dimension N (this one is given in terms of degrees of any basis elements of V), one has to check for a basis $\{\alpha_1, \ldots, \alpha_m\}$ of $H^N(\Lambda V, \delta)$ (which is finite dimensional since $\dim(V) < \infty$). The NP-hard character of the problem into question, as it is proven by L. Lechuga and A. Murillo (cf [12]), sits in the determination of the unique $j \in \{1, \ldots, m\}$ for which a represent cocycle ω_{0j} of α_j survives to reach the E_∞ term in the spectral sequence (2).

2. Basic facts

We recall here some basic facts and notation we shall need.

A simply connected space X is called *rationally elliptic* if $\dim H^*(X, \mathbb{Q}) < \infty$ and $\dim(X) \otimes \mathbb{Q} < \infty$.

A commutative graded algebra H is said to have *formal dimension* N if $H^p = 0$ for all $p > N$, and $H^N \neq 0$. An element $0 \neq \omega \in H^N$ is called a *fundamental class*.

A Sullivan algebra ([3]) is a free commutative differential graded algebra (cdga for short) $(\Lambda V$, d) (where $\Lambda V = \text{Exterior}(V^{\text{odd}}) \otimes \text{Symmetric}(V^{\text{even}})$) generated by the graded $\mathbb{K}$-vector space $V = \bigoplus_{i=0}^{i=\infty} V^i$ which has a well ordered basis $\{x_\alpha\}$ such that $dx_\alpha \in \Lambda V_{<\alpha}$. Such algebra is said minimal if $\deg(x_\alpha) < \deg(x_\beta)$ implies $\alpha < \beta$. If $V^0 = V^1 = 0$ this is equivalent to saying that $d(V) \subseteq \bigoplus_{i=2}^{i=\infty} \Lambda^i V$.

A Sullivan model ([3]) for a commutative differential graded algebra (A,d) is a quasi-isomorphism (morphism inducing isomorphism in cohomology) $(\Lambda V, d) \longrightarrow (A, d)$ with source, a Sullivan algebra. If $H^0(A) = K$, $H^1(A) = 0$ and $\dim(H^i(A, d)) < \infty$ for all $i \geq 0$, then, [6, Th.7.1], this minimal model exists. If X is a topological space any minimal model of the polynomial differential forms on X, $A_{PL}(X)$, is said a Sullivan minimal model of X.

$(\Lambda V, d)$ (or X) is said *elliptic*, if both V and $H^*(\Lambda V, d)$ are finite dimensional graded vector spaces (see for example [3]).

A Sullivan minimal model $(\Lambda V, d)$ is said to be pure if $d(V^{\text{even}}) = 0$ and $d(V^{\text{odd}}) \subset \Lambda V^{\text{even}}$. For such one, A. Murillo [9] gave an expression of a cocycle representing the fundamental class of $H(\Lambda V, d)$ in the case where $(\Lambda V, d)$ is elliptic. We recall this expression here:

Assume $\dim V < \infty$, choose homogeneous basis $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_m\}$

of V^{even} and V^{odd} respectively, and write

$$dy_j = a_j^1 x_1 + a_j^2 x_2 + \cdots + a_j^{n-1} x_{n-1} + a_j^n x_n, \quad j = 1, 2, \ldots, m,$$

where each a_j^i is a polynomial in the variables $x_i, x_{i+1}, \ldots, x_n$, and consider the matrix,

$$A = \begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^n \\ \vdots & \vdots & & \vdots \\ a_m^1 & a_m^2 & \cdots & a_m^n \end{pmatrix}.$$

For any $1 \leq j_1 < \cdots < j_n \leq m$, denote by $P_{j_1 \ldots j_n}$ the determinant of the matrix of order n formed by the columns $i_1, i_2, \ldots, i_n$ of A:

$$\begin{pmatrix} a_{j_1}^1 & \cdots & a_{j_1}^n \\ \vdots & \ddots & \vdots \\ a_{j_n}^1 & \cdots & a_{j_n}^n \end{pmatrix}.$$

Then (see [9]) if $\dim H^*(\Lambda V, d) < \infty$, the element $\omega \in \Lambda V$,

$$\omega = \sum_{1 \leq j_1 < \cdots < j_n \leq m} (-1)^{j_1 + \cdots + j_n} P_{j_1 \ldots j_n} y_1 \ldots \hat{y}_{j_1} \ldots \hat{y}_{j_n} \ldots y_m, \tag{3}$$

is a cocycle representing the fundamental class of the cohomology algebra.

3. Our spectral sequence

Let $(\Lambda V, d)$ be a Sullivan minimal model, where $d = \sum_{i \geq k} d_i$ with $d_i(V) \subseteq \Lambda^i V$ and $k \geq 2$. We first recall the filtration given in the introduction:

$$F^p = \Lambda^{\geq (k-1)p} V = \bigoplus_{i=(k-1)p}^{\infty} \Lambda^i V. \tag{4}$$

F^p is preserved by the differential d and satisfies $F^p(\Lambda V) \otimes F^q(\Lambda V) \subseteq F^{p+q}(\Lambda V)$, $\forall p, q \geq 0$, so it is a filtration of differential graded algebras. Also, since

$F^0 = \Lambda V$ and $F^{p+1} \subseteq F^p$ this filtration is decreasing and bounded, so it induces a convergent spectral sequence. Its 0^{th}-term is

$$E_0^{p,q} = \left(\frac{F^p}{F^{p+1}}\right)^{p+q} = \left(\frac{\Lambda^{\geq(k-1)p}V}{\Lambda^{\geq(k-1)(p+1)}V}\right)^{p+q}.$$

Hence, we have the identification:

$$E_0^{p,q} = \left(\Lambda^{p(k-1)}V \oplus \Lambda^{p(k-1)+1}V \oplus \cdots \oplus \Lambda^{p(k-1)+k-2}V\right)^{p+q}, \tag{5}$$

with the product given by:

$$(u_0, u_1, \ldots, u_{k-2}) \otimes (u_0^{'}, u_1^{'}, \ldots, u_{k-2}^{'}) = (v_0, v_1, \ldots, v_{k-2})$$

for all $(u_0, u_1, \ldots, u_{k-2})$, $(u_0^{'}, u_1^{'}, \ldots, u_{k-2}^{'}) \in E_0^{p,q}$ with $v_m = \sum_{i+j=m} u_i u_j^{'}$ and $m = 0, \ldots, k-2$.

The differential on E_0 is zero, hence $E_1^{p,q} = E_0^{p,q}$ and so the identification above gives the following diagram:

$$\begin{tikzcd}
E_1^{p,q} \arrow[r, "\cong"] \arrow[d, "\delta"'] & \left(\Lambda^{(k-1)p}V \oplus \Lambda^{(k-1)p+1}V \oplus \cdots \oplus \Lambda^{(k-1)p+k-2}V\right)^{p+q} \arrow[d, "\delta"] \arrow[d, dashed, "d_k,\ d_{k+1},\ \ldots,\ d_{2(k-1)-1},\ d_{2(k-1)}"'] \\
E_1^{p+1,q} \arrow[r, "\cong"] & \left(\Lambda^{(k-1)(p+1)}V \oplus \Lambda^{(k-1)(p+1)+1}V \oplus \cdots \oplus \Lambda^{(k-1)(p+1)+k-2}V\right)^{p+q+1}
\end{tikzcd}$$

with δ defined as follows,

$$\delta(u_0, u_1, \ldots, u_{k-2}) = (w_k, w_{k+1}, \ldots, w_{2k-2}) \quad \text{with} \quad w_{k+j} = \sum_{\substack{i+i'=j \\ i'=0,\ldots,k-2}} d_{k+i} u_{i'}.$$

Let $E_1^p = E_1^{p,*} = \bigoplus_{q\geq 0} E_1^{p,q}$ and $E_1^* = \bigoplus_{p\geq 0} E_1^{p,*} = \Lambda V$ as a graded vector space. In this general situation, the 1^{st}-term is the graded algebra ΛV provided with a differential δ, which is not necessarily a derivation on the set V of generators. That is, $(\Lambda V, \delta)$ is a commutative differential graded algebra, but it is not a Sullivan algebra. This gives, consequently, our spectral sequence:

$$E_2^{p,q} = H^{p,q}(\Lambda V, \delta) \Rightarrow H^{p+q}(\Lambda V, d). \tag{6}$$

Once more, using this spectral sequence, the algorithm completed by proves of claims that will appear, will give the appropriate generating class of $H^N(\Lambda V, \delta)$ that survives to the ∞ term. Accordingly, the explicit formula of LS category for this general case, is expressed in terms of the greater filtering degree of a represent of this class.

4. Proof of the main results

4.1. Proof of Theorem 1. Recall that $(\Lambda V, d)$ is assumed elliptic, so that, $\mathrm{cat}(\Lambda V, d) = e_0(\Lambda V, d)$ [4]. Notice also that the subsequent notations imposed us sometimes to replace a sum by some tuple and vice-versa.

4.1.1. The first inequality. In what follows, we put:

$$r = \sup\{s \geq 0,\ \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^{\geq s} V\}.$$

Denote by p the least integer such that $p(k-1) \leq r < (p+1)(k-1)$ and let then $\omega_0 \in \Lambda^{\geq r} V$. We have

$$\begin{aligned}\omega_0 \in\ & (\Lambda^{(k-1)p} V \oplus \cdots \oplus \Lambda^{(k-1)p+k-2} V)\\ & \oplus (\Lambda^{(k-1)p+k-1} V \oplus \cdots \oplus \Lambda^{(k-1)p+2k-3} V)\\ & \oplus \cdots\end{aligned}$$

Since $|\omega_0| = N$ and $\dim V < \infty$, there is an integer l such that

$$\omega_0 = \omega_0^0 + \omega_0^1 + \cdots + \omega_0^l$$

with $\omega_0^0 \neq 0$ and $\forall i = 0, \ldots, l$,

$$\omega_0^i = (\omega_0^{i,0}, \omega_0^{i,1}, \ldots, \omega_0^{i,k-2}) \in \Lambda^{(k-1)(p+i)} V \oplus \cdots \oplus \Lambda^{(k-1)(p+i)+k-2} V.$$

We have successively:

$$\begin{aligned}\delta(\omega_0^i) &= \delta\Big(\omega_0^{i,0}, \omega_0^{i,1}, \ldots, \omega_0^{i,k-2}\Big)\\ &= \left(d_k \omega_0^{i,0}, \sum_{i'+i''=1} d_{k+i'} \omega_0^{i,i''}, \sum_{i'+i''=2} d_{k+i'} \omega_0^{i,i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'} \omega_0^{i,i''}\right),\end{aligned}$$

$$\begin{aligned}\delta(\omega_0) &= \sum_{i=0}^{l} \delta\Big(\omega_0^{i,0}, \omega_0^{i,1}, \ldots, \omega_0^{i,k-2}\Big)\\ &= \sum_{i=0}^{l} \Bigg(d_k \omega_0^{i,0}, \sum_{i'+i''=1} d_{k+i'} \omega_0^{i,i''}, \sum_{i'+i''=2} d_{k+i'} \omega_0^{i,i''}, \ldots,\\ &\qquad \sum_{i'+i''=k-2} d_{k+i'} \omega_0^{i,i''}\Bigg)\end{aligned}$$

Also, we have $d\omega_0 = d\omega_0^0 + d\omega_0^1 + \cdots + d\omega_0^l$, with:

$$\begin{aligned}
d\omega_0^0 &= d\left(\omega_0^{0,0}, \omega_0^{0,1}, \ldots, \omega_0^{0,k-2}\right) \\
&= \left(d_k\omega_0^{0,0}, \sum_{i'+i''=1} d_{k+i'}\omega_0^{0,i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'}\omega_0^{0,i''}\right) + \cdots \\
&\in \left(\bigoplus_{k'=k-1}^{2k-3} \Lambda^{(k-1)p+k'}V\right) \oplus \cdots
\end{aligned}$$

$$\begin{aligned}
d\omega_0^1 &= d\left(\omega_0^{1,0}, \omega_0^{1,1}, \ldots, \omega_0^{1,k-2}\right) \\
&= \left(d_k\omega_0^{1,0}, \sum_{i'+i''=1} d_{k+i'}\omega_0^{1,i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'}\omega_0^{1,i''}\right) + \cdots \\
&\in \left(\bigoplus_{k'=2k-2}^{3k-4} \Lambda^{(k-1)p+k'}V\right) \oplus \cdots
\end{aligned}$$

$\cdots$

$$\begin{aligned}
d\omega_0^i &= d\left(\omega_0^{i,0}, \omega_0^{i,1}, \ldots, \omega_0^{i,k-2}\right) \\
&= \left(d_k\omega_0^{i,0}, \sum_{i'+i''=1} d_{k+i'}\omega_0^{i,i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'}\omega_0^{i,i''}\right) + \cdots \\
&\in \left(\bigoplus_{k'=(k-1)p+(i+1)k-(i+1)}^{(k-1)p+(i+2)k-(i+3)} \Lambda^{(k-1)p+k'}V\right) \oplus \cdots
\end{aligned}$$

Therefore

$$\begin{aligned}
d\omega_0 = \sum_{i=0}^{l} &\left(d_k\omega_0^{i,0}, \sum_{i'+i''=1} d_{k+i'}\omega_0^{i,i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'}\omega_0^{i,i''}\right) \\
&+ \sum_{i=0}^{l} \Big(d_{2k-2}\omega_0^{i,1} + (d_{2k-2} + d_{2k-3})\omega_0^{i,2} + \cdots + (d_{2k-2} + d_{2k-3} + \cdots \\
&\qquad + d_{k+1})\omega_0^{i,k-2}\Big) + \sum_{k'>2k-2} d_{k'}\omega_0
\end{aligned}$$

that is:

$$d\omega_0 = \delta(\omega_0) + \sum_{i=0}^{l}\Big(d_{2k-2}\omega_0^{i,1} + (d_{2k-2}+d_{2k-3})\omega_0^{i,2} + \cdots + (d_{2k-2}+\dots + d_{k+1})\omega_0^{i,k-2}\Big) + \sum_{k'>2k-2} d_{k'}\omega_0.$$

As $\delta(\omega_0)=0$, we can rewrite:

$$d\omega_0 = a_2^0 + a_3^0 + \cdots + a_{t+l}^0 \quad \text{with} \quad a_i^0 = (a_i^{0,0},\dots,a_i^{0,k-2}) \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+i)+j}V.$$

Note also that t is a fixed integer. Indeed, the degree of a_{t+l}^0 is greater than or equal to $2\big((k-1)(p+t+l)+k-2\big)$ and it coincides with $N+1$, N being the formal dimension of $(\Lambda V,d)$.
Then

$$N+1 \geq 2\big((k-1)(p+t+l)+k-2\big).$$

Hence

$$t \leq \frac{1}{2(k-1)}\big(N-2(k-1)(p+l)+5-2k\big).$$

In what follows, we take t the largest integer satisfying this inequality.

Now, we have:

$$\begin{aligned} d^2\omega_0 &= da_2^0 + da_3^0 + \cdots + da_{t+l}^0 \\ &= d(a_2^{0,0},a_2^{0,1},\dots,a_2^{0,k-2}) + d(a_3^{0,0},a_3^{0,1},\dots,a_3^{0,k-2}) + \cdots \\ &\quad + d(a_{t+l}^{0,0},a_{t+l}^{0,1},\dots,a_{t+l}^{0,k-2}), \end{aligned}$$

with

$$\begin{aligned} d(a_2^{0,0},a_2^{0,1},\dots,a_2^{0,k-2}) &= d_k(a_2^{0,0},a_2^{0,1},\dots,a_2^{0,k-2}) \\ &\quad + d_{k+1}(a_2^{0,0},a_2^{0,1},\dots,a_2^{0,k-2}) + \cdots \\ &= \Bigg(d_k a_2^{0,0}, \sum_{i'+i''=1} d_{k+i'}a_2^{0,i''},\dots,\sum_{i'+i''=k-2} d_{k+i'}a_2^{0,i''}\Bigg) \\ &\quad + \Big(d_{2k-1}a_2^{0,0} + d_{2k-2}a_2^{0,1} + \cdots,\dots\Big) + \cdots \end{aligned}$$

$$
\begin{aligned}
d(a_3^{0,0}, a_3^{0,1}, \dots, a_3^{0,k-2}) &= d_k(a_3^{0,0}, a_3^{0,1}, \dots, a_3^{0,k-2}) \\
&\quad + d_{k+1}(a_3^{0,0}, a_3^{0,1}, \dots, a_3^{0,k-2}) + \cdots \\
&= \left(d_k a_3^{0,0}, \sum_{i'+i''=1} d_{k+i'} a_3^{0,i''}, \dots, \sum_{i'+i''=k-2} d_{k+i'} a_3^{0,i''} \right) \\
&\quad + \Big(d_{2k-1} a_3^{0,0} + d_{2k-2} a_3^{0,1} + \cdots, \dots \Big) + \cdots
\end{aligned}
$$

$$\cdots$$

It follows that:

$$
\begin{aligned}
d^2\omega_0 = & \left(d_k a_2^{0,0}, \sum_{i'+i''=1} d_{k+i'} a_2^{0,i''}, \dots, \sum_{i'+i''=k-2} d_{k+i'} a_2^{0,i''} \right) \\
& + \Big(d_{2k-1} a_2^{0,0} + d_{2k-2} a_2^{0,1} + \cdots, \dots \Big) + \cdots \\
& + \Big(d_{2k-1} a_3^{0,0} + d_{2k-2} a_3^{0,1} + \cdots, \dots \Big) + \cdots
\end{aligned}
$$

Since $d^2\omega_0 = 0$, we have

$$\left(d_k a_2^{0,0}, \sum_{i'+i''=1} d_{k+i'} a_2^{0,i''}, \dots, \sum_{i'+i''=k-2} d_{k+i'} a_2^{0,i''} \right) = \delta(a_2^0) = 0$$

with $a_2^0 = (a_2^{0,0}, \dots, a_2^{0,k-2}) \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)+j} V$. Consequently a_2^0 is a δ-cocycle.

Claim 1. a_2^0 is an δ-coboundary.

Proof. Recall first that the general r^{th}-term of the spectral sequence (6) is given by the formula:

$$E_r^{p,q} = Z_r^{p,q} / Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q},$$

where

$$Z_r^{p,q} = \big\{ x \in [F^p(\Lambda V)]^{p+q} \mid dx \in [F^{p+r}(\Lambda V)]^{p+q+1} \big\}$$

and

$$B_r^{p,q} = d\big([F^{p-r}(\Lambda V)]^{p+q-1} \big) \cap F^p(\Lambda V) = d\big(Z_{r-1}^{p-r+1,q+r-2} \big).$$

Recall also that the differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ in $E_r^{*,*}$ is induced from the differential d of $(\Lambda V, d)$ by the formula $d_r([v]_r) = [dv]_r$, v being any representative in $Z_r^{p,q}$ of the class $[v]_r$ in $E_r^{p,q}$.

We still assume that $\dim H^N(\Lambda V, \delta) = 1$ and adopt notations of § 4.1.1.

Notice then $\omega_0 \in Z_2^{p,q}$ and it represents a non-zero class $[\omega_0]_2$ in $E_2^{p,q}$. Otherwise $\omega_0 = \omega_0' + d(\omega_0'')$, where $\omega_0' \in Z_1^{p+1,q-1}$ and $\omega_0'' \in B_1^{p,q}$, so that $\alpha = [\omega_0] = [\omega_0' - (d-\delta)(\omega_0'')]$. But $\omega_0' - (d-\delta)(\omega_0'') \in \Lambda^{\geq r+1}$ is a contradiction to the definition of ω_0. Now, using the isomorphism $E_2^{*,*} \cong H^{*,*}(\Lambda V, \delta)$, we deduce that, $[\omega_0]_2 \in E_2^{p,q}$ (being the only generating element) must survive to $E_3^{p,q}$, otherwise, the spectral sequence fails to converge. Whence $d_2([\omega_0]_2) = [a_2^0]_2 = 0$ in $E_2^{p+2,q-1}$, i.e., $a_2^0 \in Z_1^{p+3,q-2} + B_1^{p+2,q-1}$. However $a_2^0 \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)+j} V$, so $a_2^0 \in B_1^{p+2,q-1}$, that is $a_2^0 = d(x)$, $x \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+1)+j} V$. By wordlength argument, we have necessary $a_2^0 = \delta(x)$, which finishes the proof of Claim 1. ∎

Notice that this is the first obstruction to $[\omega_0]$ to represent a non zero class in the term $E_3^{*,*}$ of (6). The others will appear progressively as one advances in the algorithm.

Let then $b_2 \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)-(k-1)+j} V$ such that $a_2^0 = \delta(b_2)$ and put $\omega_1 = \omega_0 - b_2$. Reconsider the previous calculation for it:

$$\begin{aligned} d\omega_1 &= d\omega_0 - db_2 \\ &= (a_2^0 + a_3^0 + \cdots + a_{t+l}^0) - (d_k b_2 + d_4 b_2 + \cdots), \end{aligned}$$

with

$$d_k b_2 = d_k(b_2^0, b_2^1, \ldots, b_2^{k-2}) = (d_k b_2^0, d_k b_2^1, \ldots, d_k b_2^{k-2}) \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)+j} V,$$

$$\begin{aligned} d_{k+1} b_2 &= d_{k+1}(b_2^0, b_2^1, \ldots, b_2^{k-2}) \\ &= (d_{k+1} b_2^0, d_{k+1} b_2^1, \ldots, d_{k+1} b_2^{k-2}) \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)+j+1} V, \end{aligned}$$

$\ldots$

This implies that

$$\begin{aligned}
d\omega_1 &= a_2^0 + a_3^0 + \cdots + a_{t+l}^0 - \left(d_k b_2^0, \sum_{i'+i''=1} d_{k+i'} b_2^{i''}, \dots, \sum_{i'+i''=k-2} d_{k+i'} b_2^{i''} \right) \\
&\quad - \big(d_{2k-1} b_2^0 + \cdots, \dots \big) \\
&= a_2^0 - \delta(b_2) + a_3^0 - \big(d_{2k-1} b_2^0 + \cdots, \dots \big) + \cdots \\
&= a_3^0 - \big(d_{2k-1} b_2^0 + \cdots, \dots \big) + \cdots,
\end{aligned}$$

and then:

$$d\omega_1 = a_3^1 + a_4^1 + \cdots + a_{t+l}^1, \quad \text{with} \quad a_i^1 \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+i)+j} V.$$

So,

$$\begin{aligned}
d^2\omega_1 &= da_3^1 + da_4^1 + \cdots + da_{t+l}^1 \\
&= \left(d_k a_3^{1,0}, \sum_{i'+i''=1} d_{k+i'} a_3^{1,i''}, \dots, \sum_{i'+i''=k-2} d_{k+i'} a_3^{1,i''} \right) \\
&\quad + \left(d_{2k-1} a_3^{1,0} + \cdots, \dots \right) + \cdots
\end{aligned}$$

Since $d^2\omega_1 = 0$, by wordlength reasons,

$$\left(d_k a_3^{1,0}, \sum_{i'+i''=1} d_{k+i'} a_3^{1,i''}, \dots, \sum_{i'+i''=k-2} d_{k+i'} a_3^{1,i''} \right) = \delta(a_3^1) = 0.$$

We claim that $a_3^1 = \delta(b_3)$ and consider $\omega_2 = \omega_1 - b_3$.

We continue this process defining inductively $\omega_j = \omega_{j-1} - b_{j+1}$, $j \leq t+l-2$ such that:

$$d\omega_j = a_{j+2}^j + a_{j+3}^j + \cdots + a_{t+l}^j, \quad \text{with} \quad a_i^j \in \bigoplus_{h=0}^{k-2} \Lambda^{(k-1)(p+i)+h} V$$

and a_{j+2}^j a δ-cocycle.

CLAIM 2. a^j_{j+2} is a δ-coboundary, i.e., there is

$$b_{j+2} \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+j+2)-(k-1)+j} V$$

such that $\delta(b_{j+2}) = a^j_{j+2}$; $1 \leq j \leq t+l-2$.

Proof. We proceed in the same manner as for the first claim. Indeed, we have clearly for any $1 \leq j \leq t+l-2$, $\omega_j = \omega_{j-1} - b_{j+1} = \omega_0 - b_2 - b_3 - \cdots - b_{j+1} \in Z^{p,q}_{j+2}$ and it represents a non zero class $[\omega_j]_{j+2}$ in $E^{p,q}_{j+2}$ which is also one dimensional. Whence as in Claim 1, we conclude that, a^j_{j+2} is a δ-coboundary for all $1 \leq j \leq t+l-2$. ∎

Consider $\omega_{t+l-1} = \omega_{t+l-2} - b_{t+l}$, where $\delta(b_{t+l}) = a^{t+l-2}_{t+l}$. Notice that $|d\omega_{t+l-1}| = |d\omega_{t+l-2}| = N+1$, but by the hypothesis on t, we have $d(\omega_{t+l-2}) = a^{t+l-2}_{t+l}$ and then

$$|d(\omega_{t+l-2} - b_{t+l})| = |a^{t+l-2}_{t+l} - \delta(b_{t+l}) - (d-\delta)b_{t+l}| = |-(d-\delta)b_{t+l}| > N+1.$$

It follows that $d\omega_{t+l-1} = 0$, that is ω_{t+l-1} is a d-cocycle. But it can't be a d-coboundary. Indeed suppose that $\omega_{t+l-1} = (\omega^0_0 + \omega^1_0 + \cdots + \omega^l_0) - (b_2 + b_3 + \cdots + b_{t+l})$, were a d-coboundary, by wordlength reasons, ω^0_0 would be a δ-coboundary, i.e., there is $x \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)p-(k-1)+j} V$ such that $\delta(x) = \omega^0_0$. Then

$$\omega_0 = \delta(x) + \omega^1_0 + \cdots + \omega^l_0.$$

Since $\delta(\omega_0) = 0$, we would have $\delta(\omega^1_0 + \cdots + \omega^l_0) = 0$ and then $[\omega_0] = [\omega^1_0 + \cdots + \omega^l_0]$. But $\omega^1_0 + \cdots + \omega^l_0 \in \Lambda^{>r} V$, contradicts the property of ω_0. Consequently ω_{t+l-1} represents the fundamental class of $(\Lambda V, d)$.

Finally, since $\omega_{t+l-1} \in \Lambda^{\geq r} V$ we have

$$e_0(\Lambda V, d) \geq r.$$

4.1.2. FOR THE SECOND INEQUALITY. Denote $s = e_0(\Lambda V, d)$ and let $\omega \in \Lambda^{\geq s} V$ be a cocycle representing the generating class α of $H^N(\Lambda V, d)$.

Write $\omega = \omega_0 + \omega_1 + \cdots + \omega_t$, $\omega_i \in \Lambda^{s+i} V$. We deduce that:

$$\begin{aligned} d\omega &= \left(d_k \omega_0 + \sum_{i+i'=1} d_{k+i} \omega_{i'} + \cdots + \sum_{i+i'=k-2} d_{k+i} \omega_{i'} \right) + d_k \omega_{k-1} + d_{2k-1} \omega_0 + \cdots \\ &= \delta(\omega_0, \omega_1, \ldots, \omega_{k-2}) + \cdots \end{aligned}$$

Since $d\omega = 0$, by wordlength reasons, it follows that $\delta(\omega_0, \omega_1, \dots, \omega_{k-2}) = 0$. If $(\omega_0, \omega_1, \dots, \omega_{k-2})$, were a δ-boundary, i.e., $(\omega_0, \omega_1, \dots, \omega_{k-2}) = \delta(x)$, then

$$\begin{aligned} \omega - dx &= (\omega_0, \omega_1, \dots, \omega_{k-2}) - \delta(x) + (\omega_{k-1} + \dots + \omega_t) - (d - \delta)(x) \\ &= (\omega_{k-1} + \dots + \omega_t) - (d - \delta)(x), \end{aligned}$$

so, $\omega - dx \in \Lambda^{\geq s+k-1}V$, which contradicts the fact $s = e_0(\Lambda V, d)$. Hence $(\omega_0, \omega_1, \dots, \omega_{k-2})$ represents the generating class of $H^N(\Lambda V, \delta)$. But $(\omega_0, \omega_1, \dots, \omega_{k-2}) \in \Lambda^{\geq s}V$ implies that $s \leq r$. Consequently, $e_0(\Lambda V, d) \leq r$.

Thus, we conclude that

$$e_0(\Lambda V, d) = r.$$

4.2. Proof of Theorem 2. It suffices to remark that since $(\Lambda V, d)$ is elliptic, it has Poincaré duality property and then $\dim H^N(\Lambda V, d) = 1$. The convergence of (6) implies that $\dim E_\infty^{*,*} = 1$. Hence there is a unique (p, q) such that $p+q = N$ and $E_\infty^{*,*} = E_\infty^{p,q}$. Consequently only one of the generating classes $\alpha_1, \dots, \alpha_m$ had to survive to E_∞. Let α_j this representative class and (p_j, q_j) its pair of degrees. ∎

Example 1. Let $d = \sum_{i\geq 3} d_i$ and $(\Lambda V, d)$ be the model defined by $V^{\text{even}} = < x_2, x'_2 >$, $V^{\text{odd}} =< y_5, y_7, y'_7 >$, $dx_2 = dx'_2 = 0$, $dy_5 = x_2^3$, $dy_7 = x'^4_2$ and $dy'_7 = x_2^2 x'^2_2$, in which subscripts denote degrees.

For $k \geq 3$, $l \geq 0$, we have

$$x_2^k x'^l_2 = x_2^{k-3} x_2^3 x'^l_2 = d\big(y_5 x_2^{k-3} x'^l_2\big).$$

For $k \geq 4$, $l \geq 0$,

$$x'^k_2 x_2^l = x_2^l x'^{k-4}_2 x'^4_2 = d\big(x_2^l x'^{k-4}_2 y_7\big).$$

Clearly we have

$$\dim H(\Lambda V, d) < \infty \text{ and } \dim H(\Lambda V, d_3) = \infty.$$

Using A. Murillo's algorithm (cf. §2) the matrix determining the fundamental class is:

$$A = \begin{pmatrix} x_2^2 & 0 \\ 0 & x'^3_2 \\ x_2 x'^2_2 & 0 \end{pmatrix},$$

so, $\omega = -x_2^2 x_2^{,3} y_7^{,} + x_2 x_2^{,5} y_5 \in \Lambda^{\geq 6} V$ is a generator of this fundamental cohomology class.

It follows that $e_0(\Lambda V, d) = 6 \neq m + n(k-2)$.

Example 2. Let $d = \sum_{i \geq 3} d_i$ and $(\Lambda V, d)$ be the model defined by $V^{\text{even}} = < x_2, x_2^{,} >$, $V^{\text{odd}} = < y_5, y_9, y_9^{,} >$, $dx_2 = dx_2^{,} = 0$, $dy_5 = x_2^3$, $dy_9 = x_2^{,5}$ and $dy_9^{,} = x_2^3 x_2^{,2}$.

Clearly we have

$$\dim H(\Lambda V, d) < \infty \text{ and } \dim H(\Lambda V, d_3) = \infty.$$

Using A. Murillo's algorithm (cf. §2) the matrix determining the fundamental class is:

$$A = \begin{pmatrix} x_2^2 & 0 \\ 0 & x_2^{,4} \\ x_2^2 x_2^{,2} & 0 \end{pmatrix},$$

so, $\omega = -x_2^2 x_2^{,4} y_9^{,} + x_2^2 x_2^{,6} y_5 \in \Lambda^{\geq 7} V$ is a generator of this fundamental cohomology class.

It follows that $e_0(\Lambda V, d) = 7 \neq m + n(k-2)$.

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